CES-Fréchet modeling of farmer choices

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Model setup

- Risk-neutral farmer/landowner facing the choice of allocating its land endowment x_1 to crops.
- Crops are indexed by $k = l \in \mathcal{K} \equiv \{1, \ldots, K\}$
- Crop production requires 2 bundles of inputs:
  1. R inputs that are partial substitutes (e.g., land, fertilizers and water), with land indexed $r = 1$, with substitution elasticity $0 < \sigma_k < 1$.
  2. Unspecified for now and corresponds to non-land value-added and is non substitutable to the first bundle.
- Land is heterogeneous and composed of a continuum of parcels indexed by ω defined over $[0, 1]$.

Production function

$$Q_k(\omega) = \min\left\{\left[A_{1,k}(\omega)(x_{1,k}(\omega))^{(\sigma_k-1)/\sigma_k} + \sum_{r=2}^{R} A_{r,k}(x_{r,k}(\omega))^{(\sigma_k-1)/\sigma_k}\right]^{\sigma_k/(\sigma_k-1)}, N_k(\omega)/\nu_k\right\},$$

- $A_{1,k}(\omega) \geq 0$ a parameter governing land productivity
- $A_{r,k} \geq 0$ with $r \neq 1$ are productivity shifters for the inputs affecting yields
- $x_{r,k}(\omega)$ is input demand,
- $N_k(\omega)$ is the value added demand,
- $\nu_k > 0$ is a productivity shifter for value added.

Price indexes

From the Leontief structure:

$$p_k = P_k^X + w\nu_k,$$

where P_k^X is the price of the first input bundle and w is the wage. From CES standard algebra,

$$P_k^X = \left[\left(A_{1,k}\left(\omega\right)\right)^{\sigma_k - 1} \left(\pi_{1,k}\left(\omega\right)\right)^{1-\sigma_k} + \sum_{r=2}^{R} A_{r,k}^{\sigma_k - 1} \pi_{r,k}^{1-\sigma_k} \right]^{1/(1-\sigma_k)}.$$

Then, we can express the land rents per hectare as:

$$\begin{aligned} \pi_{1,k}\left(\omega\right) &= A_{1,k}\left(\omega\right) \left[\left(P_k^X\right)^{1-\sigma_k} - \sum_{r=2}^{R} A_{r,k}^{\sigma_k - 1} \pi_{r,k}^{1-\sigma_k} \right]^{1/(1-\sigma_k)}, \\ &= A_{1,k}\left(\omega\right) \underbrace{\left[\left(p_k - w\nu_k\right)^{1-\sigma_{ku}} - \sum_{r=2}^{R} A_{r,k}^{\sigma_k - 1} \pi_{r,k}^{1-\sigma_k} \right]^{1/(1-\sigma_k)}}_{=r_k}. \end{aligned}$$

Fréchet assumption

$A_{1,k}(\omega)$ are i.i.d. from a Fréchet distribution with shape $\theta > 1$ and scale $\gamma A_{1,k} > 0$:

$$\Pr\left(A_{1,k}(\omega) \leq a\right) = \exp\left[-\left(\frac{a}{\gamma A_{1,k}}\right)^{-\theta}\right] \quad \forall a \in \mathbb{R}_{>0}.$$

- $\gamma \equiv \left(\Gamma\left(1 - 1/\theta\right)\right)^{-1}$ a scaling parameter introduced so that $A_{1,k}$ is the unconditional productivity of land, $A_{1,k} = \mathrm{E}\left[A_{1,k}(\omega)\right]$, the productivity if all the land was planted with crop k.
- θ measures the dispersion of yields around their average $A_{1,k}$: a higher θ indicates more homogeneity and a lower θ more heterogeneity.

It follows that $\pi_{1,k}(\omega)$ is distributed Fréchet with parameters θ and $\gamma r_k A_{1,k}$

Crop choices

The probability that crop k is the most profitable on parcel ω is defined by

$$\lambda_k = \Pr\left(\pi_{1,k}(\omega) \in \arg\max_{l \in \mathcal{K}} \pi_{1,l}(\omega)\right),$$

$=$ also the share of land allocated to k.

$$\lambda_k = \frac{\pi_{1,k}^{\theta}}{\sum_{l \in \mathcal{K}} \pi_{1,l}^{\theta}},$$

where $\pi_{1,k} \equiv r_k A_{1,k}$ denotes the unconditional land rents if all the land is planted with crop k.

Crop production

From CES and Fréchet algebra:

$$Q_k = x_1 \lambda_k \left(\frac{r_k}{P_k^X} \right)^{\sigma_k} \mathsf{E} \left[A_{1,k}(\omega) \left| \pi_{1,k}(\omega) \in \arg\max_{l \in \mathcal{K}} \pi_{1,l}(\omega) \right. \right],$$

$$= x_1 A_{1,k} \lambda_k^{(\theta-1)/\theta} \left(\frac{r_k}{P_k^X} \right)^{\sigma_k}.$$

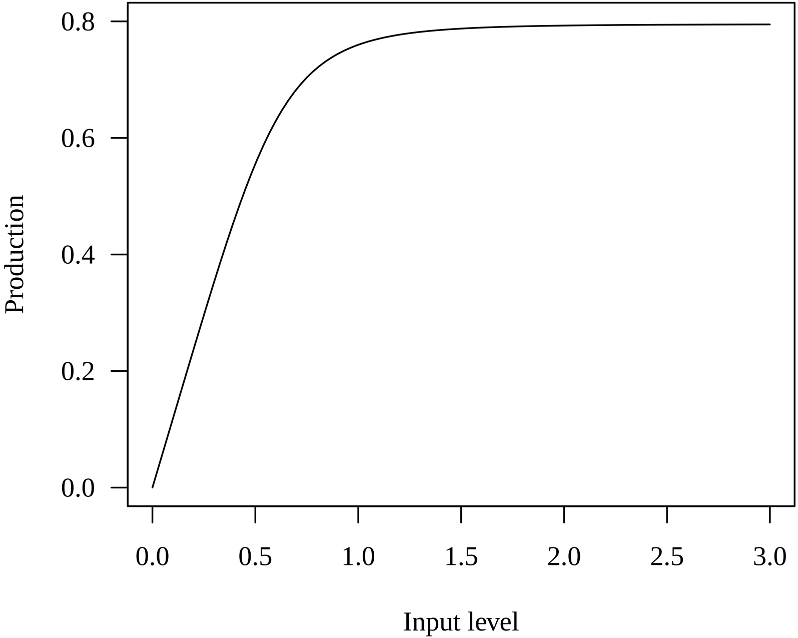
Input demand

$$x_{r,k} = \mathsf{E}\left(x_{r,k}\left(\omega\right) \left| \pi_{1,k}\left(\omega\right) \in \arg\max_{l \in K} \pi_{1,l}\left(\omega\right) \right.\right),$$

which gives

$$\begin{aligned} x_{r,k} &= \mathsf{E}\left(A_{r,k}^{\sigma_k} Q_k\left(\omega\right) \left(\frac{\pi_{r,k}}{P_k^X}\right)^{-\sigma_k} \left| \pi_{1,k}\left(\omega\right) \in \arg\max_{l \in K} \pi_{1,l}\left(\omega\right) \right.\right), \\ &= A_{r,k}^{\sigma_k} \left(\frac{\pi_{r,k}}{P_k^X}\right)^{-\sigma_k} \mathsf{E}\left(Q_k\left(\omega\right) \left| \pi_{1,k}\left(\omega\right) \in \arg\max_{l \in K} \pi_{1,l}\left(\omega\right) \right.\right), \\ &= A_{r,k}^{\sigma_k} \left(\frac{\pi_{r,k}}{P_k^X}\right)^{-\sigma_k} Q_k. \end{aligned}$$

Parcel-level production function



In exact hat algebra

$$\hat{\lambda}_k : \hat{\lambda}_k = \frac{\left(\hat{A}_{1,k}\hat{r}_k\right)^{\theta}}{\sum_{l\in\mathcal{K}} \lambda_l \left(\hat{A}_{1,l}\hat{r}_l\right)^{\theta}},$$

$$\hat{x}_{r,k} : \hat{x}_{r,k} = \left(\frac{\hat{\pi}_{r,k}}{\hat{P}_k^X}\right)^{-\sigma_k} \hat{Q}_k, \text{ for } r \geq 2$$

$$\hat{Q}_k : \hat{Q}_k = \hat{A}_{1,k}\hat{\lambda}_k^{(\theta-1)/\theta} \left(\frac{\hat{r}_k}{\hat{P}_k^X}\right)^{\sigma_k},$$

$$\hat{P}_k^X : \alpha_k^X \hat{P}_k^X = \hat{p}_k - \left(1-\alpha_k^X\right)\hat{w}_k,$$

$$\hat{r}_k : \hat{P}_k^X = \left(\alpha_{1,k}^X \hat{r}_k^{1-\sigma_k} + \sum_{r=2}^{R} \alpha_{r,k}^X \hat{\pi}_{r,k}^{1-\sigma_k}\right)^{1/(1-\sigma_k)},$$

3 extensions

- Multiple fields
- Non zero production at zero input
- More flexible acreage elasticities (not here).

Multiple fields

- There are $f \in 1, \dots, F$ fields that are heterogeneous in their productivity. Fields can be defined on a grid or on land classes (GAEZ).
- There are no transport costs between fields, so that they all face the same prices, and labor productivity shifters ν_k are the same.

New equations

$$
\begin{aligned}
Q_k &: Q_k = \sum_{f=1}^{F} \overbrace{x_1^f A_{1,k}^f \left(\lambda_k^f\right)^{(\theta-1)/\theta} \left(\frac{r_k^f}{P_k^X}\right)^{\sigma_k}}^{=Q_k^f}, \\
\lambda_k^f &: \lambda_k^f = \frac{\left(A_{1,k}^f r_k^f\right)^\theta}{\sum_{l\in\mathcal{K}} \left(A_{1,l}^f r_l^f\right)^\theta}, \\
r_k^f &: P_k^X = \left[\left(r_k^f\right)^{1-\sigma_k} + \sum_{r=2}^{R} \left(A_{r,k}^f\right)^{\sigma_k-1} \pi_{r,k}^{1-\sigma_k}\right]^{1/(1-\sigma_k)}, \\
P_k^X &: p_k = P_k^X + w\nu_k, \\
x_{r,k} &: x_{r,k} = \left(\frac{\pi_{r,k}}{P_k^X}\right)^{\sigma_k} \sum_{f=1}^{F} \left(A_{r,k}^f\right)^{\sigma_k} Q_k^f.
\end{aligned}
$$

Elasticities

$$\frac{\partial \ln Q_k}{\partial \ln p_k} = \sum_{f=1}^{F} \frac{Q_k^f}{Q_k} \frac{1}{\alpha_k^X \alpha_{1,k}^{X,f}} \left[(\theta - 1) \left(1 - \lambda_k^f\right) + \sigma_k \left(1 - \alpha_{1,k}^{X,f}\right) \right].$$

Non zero production at zero input

Each crop can be produced using two technology: a CES technology and a no-input technology (except value added). Let's use $\tilde{x}$ for the variable x under the CES technology and $\check{x}$ for the no-input one. Let's assume that when produced under these two technologies, the same crop has different productivity distribution with the following cumulative distribution:

$$F\left(a\right)=\exp\left\{-\sum_{k\in K}\left[\left(\frac{\tilde{a}_k}{\gamma\tilde{A}_{1,k}}\right)^{-\theta/(1-\rho_k)}+\left(\frac{\check{a}_k}{\gamma\check{A}_{1,k}}\right)^{-\theta/(1-\rho_k)}\right]^{1-\rho_k}\right\},$$

where ρ_k parameterizes the correlation between the two technology.

New equations

$$Q_k = x_1 \lambda_k^{(\theta-1)/\theta} \left[\tilde{A}_{1,k} \left(\frac{\tilde{r}_k}{P_k^X} \right)^{\sigma_k} \tilde{\lambda}_k^{(\theta-1+\rho_k)/\theta} + \check{A}_{1,k} \check{\lambda}_k^{(\theta-1+\rho_k)/\theta} \right], \quad (1)$$

where

$$1 = \tilde{\lambda}_k + \check{\lambda}_k, \quad (2)$$

$$\tilde{\lambda}_k = \frac{\left(\tilde{A}_{1,k} \tilde{r}_k \right)^{\theta/(1-\rho_k)}}{\left(\tilde{A}_{1,k} \tilde{r}_k \right)^{\theta/(1-\rho_k)} + \left(\check{A}_{1,k} \check{r}_k \right)^{\theta/(1-\rho_k)}}, \quad (3)$$

$$\lambda_k = \frac{\left[\left(\tilde{A}_{1,k} \tilde{r}_k \right)^{\theta/(1-\rho_k)} + \left(\check{A}_{1,k} \check{r}_k \right)^{\theta/(1-\rho_k)} \right]^{1-\rho_k}}{\sum_{l \in K} \left[\left(\tilde{A}_{1,l} \tilde{r}_l \right)^{\theta/(1-\rho_l)} + \left(\check{A}_{1,l} \check{r}_l \right)^{\theta/(1-\rho_l)} \right]^{1-\rho_l}}, \quad (4)$$

$$\check{r}_k = p_k - w \check{\nu}_k \quad (5)$$

Data

- Share of land revenues or Acreage share (for λ_k^f)
- Acreage or supply elasticities (θ)
- σ_k: yield elasticities or fertilizer response function.
- α_k^X share of land and other inputs in production costs.
- $\alpha_{r,k}^{X,f}$ share of each input in the bundle or in a biophysical approach the input levels.
- Q_k^f/Q_k or $A_{1,k}^f$
- Others